Towards the reconstruction of poset

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Rapports de Recherche

1992

25ème anniversaire

N° 1660

Programme 1
Architectures parallèles, Bases de données, Réseaux et Systèmes distribués

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Avril 1992

RR.1660

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Towards the Reconstruction of Poset [1]

Publication Interne n°640, Mars 1992, 22 pages, Programme 1

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abstract

The reconstruction conjecture for posets is the following: "Every finite poset P of more than three elements is uniquely determined—up to isomorphism—by its collection of (unlabelled) one-element-deleted subposets $\langle P - \{x\} : x \in V(P)\rangle$". This conjecture belongs to the list of open problems in *Order*. We show that disconnected posets, posets with unique minimal (respectively, maximal) element and interval orders are reconstructible and that N-free orders are recognizable. We show that the following parameters are reconstructible: the number of minimal (respectively, maximal) elements, the level structure, the ideal-size sequence of the maximal elements, the ideal-size (respectively, filter-size) sequence of any fixed level of the HASSE-diagram and the number of edges of the HASSE-diagram. This is considered to be a first step towards a proof of the reconstruction conjecture for posets.

Vers une Reconstruction des Ensembles Ordonnés

résumé

Parmi les problèmes ouverts du journal Order, apparaît la conjecture suivante sur la reconstructibilité des ensembles ordonnés : "Tout ensemble ordonné fini P, ayant plus de trois éléments, est uniquement déterminé —à isomorphisme près— par la collection de ses restrictions aux parties n'ayant qu'un seul élément de moins : $\langle P - \{x\} : x \in V(P)\rangle$." Dans l'optique de cette conjecture, nous montrons d'une part que les ensembles ordonnés non connexes, ceux ayant un plus petit (resp. plus grand) élément et ceux dits d'intervalles sont reconstructibles et nous montrons d'autre part que les ensembles ordonnés sans-N sont reconnaissables. Nous montrons également que les paramètres suivants sont reconstructibles : le nombre d'éléments minimaux (resp. maximaux), la décomposition en rangs, la séquence des cardinaux des idéaux des éléments maximaux, la séquence des cardinaux des idéaux (resp. filtres) de chacun des rangs et le nombre d'arêtes du graphe de HASSE.

[1] Research partly supported by DAAD

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Towards the Reconstruction of Poset

Contents

1 Introduction

Our research was initialized by a very interesting note of Sands in the problem session of *Order* [25]:

> The reconstruction conjecture for graphs has a vast literature - even (since 1980) its own subsection in *Mathematical Reviews*. See the surveys by Bondy and Hemminger [4] and Nash-Williams [20] for the history of this problem.
>
> The analogous conjecture for directed graphs has been less extensively studied, due only partly to the fact that it is false; counterexamples of arbitrarily large finite order have been found by Stockmeyer [22].
>
> A special case which is still open, and on which virtually nothing has been published, is the subject of this note: poset reconstruction.
>
> *Every finite partially ordered set P of more than three elements is—up to isomorphism—uniquely determined by its collection $\{P - \{x\} : x \in V(P)\}$ of (unlabelled) one-point-deleted "subposets".*
>
> ..., S.K. Das, has found the only nontrivial results on it to date. In his thesis [9] he proved that posets of more than three elements whose HASSE-diagrams are trees are reconstructible. The proof, unfortunately, has not been published and is apparently very long. He also found some reconstructible properties; for instance he proved that the number of elements of height i in a poset P can be determined from the $P - \{x\}$'s (see Proposition 1 of [8] for a statement of his result).
>
> ..., another paper relevant to poset reconstruction is by S. Hyyrö [14], in which it is shown that certain kinds of bipartite posets are reconstructible.

S.K. Das already mentioned that the condition $|V(P)| > 3$ is necessary because two nonisomorphic posets with three elements give rise to the same collection of "subposets" (see figure 1).

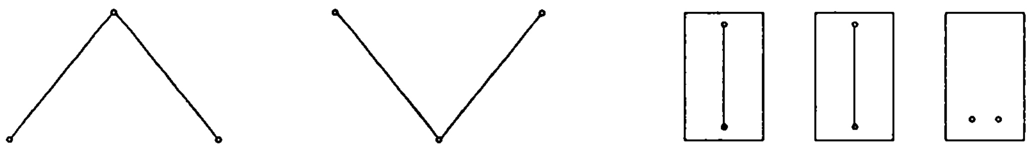

Figure 1: Two non-isomorphic posets with the same deck

Since we was not able to get the original Thesis of Das and since these results are important for us, we shall give our own proof of some of Das' Theorems.
The poset reconstruction conjecture arises from the graph reconstruction conjecture. Harary's survey of the reconstruction conjecture recounts the origins of the problem [13] :

> The author first heard of this fascinating problem when Kelly [15] proved the theorem for trees in 1957. This result was obtained in Kelly's doctoral dissertation which was written under Ulam, who published [24] a statement of the problem in 1960 (although it was already known to him in 1929, when he assiduously collected mathematical problems posed by his fellow graduate students and professors in Lwów, Poland.)

Therefore Harary proposed to call the problem the reconstruction conjecture, nowadays better the graph (vertex) reconstruction conjecture. The reconstruction conjecture states:

Any graph with at least three vertices can be reconstructed —up to isomorphism—from its collection of one-vertex-deleted subgraphs.
Also Harary gave a nice and useful reformulation of the graph reconstruction conjecture [11]:
Somebody draws on cards all one-vertex-deleted graphs of an unknown graph G, one subgraph per card. Can we reconstruct the original graph from this deck of cards, up to isomorphism?

This problem is widely believed as one of the most interesting and challenging open problems in graph theory.

Recently, Bondy gave a survey talk on the graph reconstruction conjecture at the Britsh Combinatorial Conference 1991, thereby a great part dealing with the so-called edge reconstruction conjecture, where the deck is created by all the one-edge-deleted subgraphs, which is known to be equivalent to the (vertex) reconstruction conjecture for line graphs [3].

The complexity of problems related to the graph reconstruction conjecture was studied in [16]. They showed that one of these problems—called **DECK CHECKING**—is log-space isomorphic to the well-known **GRAPH ISOMORPHISM** problem, i.e. they are almost the same problems from complexity point of view.

Regarding reconstruction two other important facts should be noticed:

1. The reconstructibility of n-ary relations has been negatively answered for every $n \geq 3$ by M. Pouzet [21].

2. The k-reconstruction problem of binary relations (i.e. the collection of all i-element-deleted subsets for every $i \in \{1, 2, \ldots, k\}$ is given) has been extensively studied in the last years: Lopez and Rauzy [17] have shown that binary relations with more than 10 elements are 4-reconstructible.

Our paper is organized as follows: Section 2 gives necessary preliminaries. In Section 3 we review the known facts from graph reconstruction theory which are useful for the poset reconstruction conjecture. In Section 4 we give a reformulation of the well-known Kelly lemma to posets, enabeling the proof of the reconstructibility of disconnected posets. Furthermore, the reconstrutibility of posets with unique minimal (respectively, maximal) element is shown. In Section 5 it is shown that the number of minimal and the number of maximal elements of a poset is reconstructible. In Section 6 it is shown that a mimimal (respectively, maximal) card can be determined from the deck and that the level-structure of a poset is reconstructible. Moreover, the ideal-size (respectively, filter-size) sequence of any fixed level of the poset is reconstructible. In Section 7 the reconstructibility of the number of edges in the HASSE-diagram is shown. In Section 8 we show that interval orders are recontructible and that N-free orders are recognizable. Finally we mention some conjectures and possible ways of ongoing research.

2 Definitions and preliminaries

We consider a partially ordered set (or *poset*) P to be a pair $(V(P), \prec_P)$, where $V(P)$ is a set and $\prec_P$ is an irreflexive, antisymmetric and transitive binary relation on $V(P)$. We call $V(P)$ the *ground set* while $\prec_P$ is said to be a *partial order* on $V(P)$.

Throughout the whole paper, we consider only finite posets with more than three elements. We now introduce some classical definitions related to a poset $P = (V(P), \prec_P)$:

Let $x, y \in V(P)$ with $x \neq y$. We say that x and y are *comparable* in P, when either $x \prec_P y$ or $y \prec_P x$ hold. On the other hand, x and y are said to be *incomparable* in P, if neither $x \prec_P y$ nor $y \prec_P x$ hold. If $x \prec_P y$ holds, then x is said to be a *predecessor* of y in P and y is said to be a *successor* of x in P.

We say that x is *covered* by y (or y *covers* x), denoted $x \multimap\!\!\prec_P y$, when $x \prec_P y$, and there is no element $z \in V(P)$ for which $x \prec_P z$ and $z \prec_P y$. Then x is said to be an *immediate predecessor* of y in P, conversely, y is said to be an *immediate successor* of x in P. This relation is the *covering relation* of P, i.e. the transitive reduction of $\prec_P$.

Furthermore, if x is an immediate predecessor of y in P such that y is its unique (immediate) successor in P, then x is said to be a ***private predecessor*** of y in P. In a similar manner, the notion ***private successor*** can be defined.

If $S \subseteq V(P)$ and $S \neq \emptyset$, then the restriction of $\prec_P$ to S, denoted $\prec_P|_S$, gives the poset $(S, \prec_P|_S)$ on S denoted $P|_S$. This poset $P|_S$ is said to be a ***subposet*** of $(V(P), \prec_P)$. When $S = V(P) - \{x\}$ for an element $x \in V(P)$, we will write $P - \{x\}$ instead of $(V(P) - \{x\}, \prec_P|_{V(P)-\{x\}})$, and $P - \{x\}$ is said to be a ***one-element-deleted subposet*** of P.

An element $x \in V(P)$ is called a *minimal* element (respectively, *maximal* element) if there is no element $y \in V(P)$ with $y \prec_P x$ (respectively, $x \prec_P y$).

P is called a ***chain*** if every pair of distinct elements $x, y \in V(P)$ is comparable. Similarly, P is called an ***antichain*** if every pair of distinct elements $x, y \in V(P)$ is incomparable. In the same way a subset S of $V(P)$ is called a chain (respectively, antichain) if the subposet $(S, \prec_P|_S)$ is a chain (respectively, antichain). The ***width*** of P, denoted $w(P)$, is the maximum cardinality of an antichain in P. The ***height*** of P, denoted $h(P)$, is one less than the maximum cardinality of a chain in P.

The ***rank*** of an element x in P, denoted $rank(x, P)$, is one less than the maximum cardinality of a chain in P with maximal element x.

The i^{th} *level* of P, denoted by $H_i(P)$, is the subset of $V(P)$ containing all elements with rank i in P: $H_i(P) = \{x \in V(P)\colon rank(x, P) = i\}$. The *level-structure* of P, is the sequence, sorted by increassing level (or rank), of the cardinatlities of each level:

$$(|H_0(P)|, |H_1(P)|, \ldots, |H_{h(P)}(P)|).$$

One usually associates to a poset P the ***comparability graph*** $G(P) = (V(P), E)$, where two vertices of the graph are joined by an edge iff the corresponding pair of elements of P is comparable. A parameter or a function is said to be a ***comparability invariant*** if it has the same value for all posets with the same comparability graph.

Another useful graph, a directed one, associated to a poset is its *covering graph*, i.e. the graph of its covering relation. Usually one prefers, in fact, a certain pictural version of this graph, namely the *HASSE-diagram*. Throughout the paper, we use the concept of the *ranked HASSE-diagram*, that we simply call *HASSE-diagram*. The (ranked) HASSE-diagram of P, denoted $H(P)$, is the covering graph of P drawn by increasing level, where all elements of a level are drawn on the same horizontal line and where the orientation of the edges is omited. Generally the drawing is from the bottom to the top of the picture as illustrated in figure 2.

A *2-transitive edge* of P, is an edge of $H(P - \{x\}) \setminus H(P)$ for some $x \in V(P)$. Thus the following situation in P is necessary for a 2-transitive edge (y, z) with $y \prec_P z$: there is an $x \in V(P)$, such that $y \multimap\!\!\prec_P x \multimap\!\!\prec_P z$ and there is no $x' \in V(P)$, such that $x' \neq x$ and $y \prec_P x' \prec_P z$ (see figure 3).

If C is an induced subgraph of the HASSE-diagram $H(P)$ of P, then the corresponding poset is said to be a *covering subposet* of P. A well-known example is the covering subposet "N" which means that we have four elements in the poset with exactly the following covering relations: $a \multimap\!\!\prec_P b$, $c \multimap\!\!\prec_P d$ and $c \multimap\!\!\prec_P b$ (see figure 4 (a)).

The *dual* of P on the set $V(P)$, denoted by P^*, has the same ground set as P and the partial order $\prec_{P^*}$, defined by $x \prec_{P^*} y$ iff $y \prec_P x$ for all $x, y \in V(P)$.

Two posets $P = (V(P), \prec_P)$ and $Q = (V(Q), \prec_Q)$ are said to be *isomorphic*, denoted by $P \cong Q$, if there exists a bijection $f : V(P) \longrightarrow V(Q)$ such that $x \prec_P y$ iff $f(x) \prec_Q f(y)$ for all $x, y \in V(P)$.

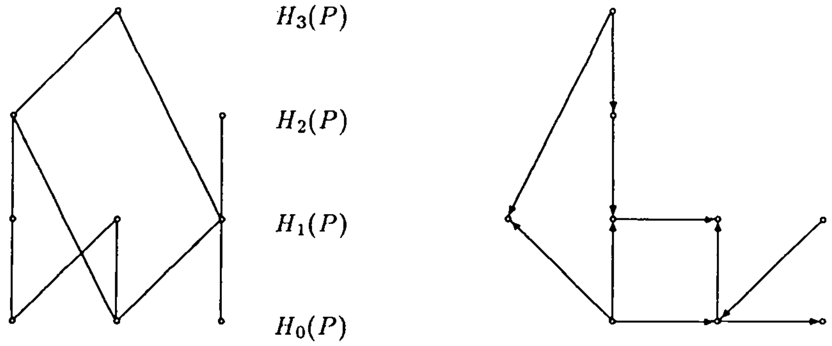

Figure 2: A (ranked) HASSE-diagram and the corresponding covering graph

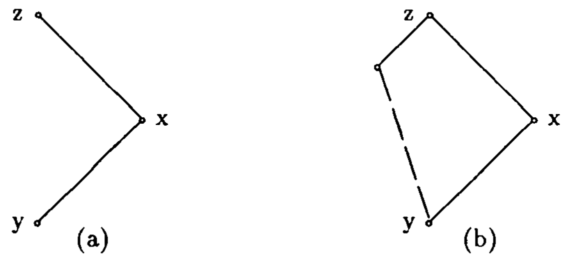

Figure 3: (a) (y, z) 2-transitive; (b) (y, z) not 2-transitive

Similar to Harary's reformulation of the graph reconstruction conjecture [11] the poset reconstruction conjecture can be formulated as follows:
Somebody draws on cards all one-element-deleted subposets of an unknown poset P (say as HASSE-diagrams), one subposet per card. Can we reconstruct the original poset from this deck of cards, up to isomorphism?

We shall mainly use this model of the reconstruction conjecture. In this sense we shall often speak about one-element-deleted subposets as *cards* and about the unknown poset as a *preimage* of the deck. Following the classical definitions in graph reconstruction [15, 24, 26], we introduce similar notions regarding poset reconstruction. Let $P = (V(P), \prec_P)$ be a poset, then:

1. The *deck* of P is the collection $\langle P - \{x\} : x \in V(P)\rangle$ of its (unlabelled) one-element-deleted subposets. As already mentioned, a subposet $P - \{x\}$ is usually said to be a card of P.

2. A poset Q is a *reconstruction* of P if there is a bijection $\sigma : V(P) \longrightarrow V(Q)$ such that for every $x \in V(P)$ holds: $P - \{x\} \cong Q - \{\sigma(x)\}$. We shall often say that P and Q have the same deck if this condition is fulfilled.

3. A parameter or a function (defined on all posets) is said to be *reconstructible* if for every poset P it has the same value for all reconstructions of P.

4. A poset P is said to be *reconstructible*, if every reconstruction of P is isomorphic to P.

5. A class C of posets is said to be *recognizable*, if any reconstruction of a poset in C belongs to C. (Note that cards might not belong to C; if the class C is not hereditary.)

6. A class $\mathcal{C}$ of posets is said to be *weak reconstructible*, if for any poset P in $\mathcal{C}$, any reconstruction of P belonging to $\mathcal{C}$ is isomorphic to P.

7. A class $\mathcal{C}$ of posets is said to be *reconstructible* if any poset in $\mathcal{C}$ is reconstructible.

Remark: A class $\mathcal{C}$ of posets, which is recognizable and weak reconstructible, is reconstructible.

Througouht the paper we often rely on the following two transformations associated to a deck, say $P_1, P_2, \ldots, P_n$:

1. Given a deck, we can easily construct the *dual deck* $P_1^*, P_2^*, \ldots, P_n^*$. Clearly, P is a preimage of $P_1, P_2, \ldots, P_n$ iff P^* is a preimage of $P_1^*, P_2^*, \ldots, P_n^*$. Consequently, P is reconstructible iff P^* is reconstructible. Furthermore, if a parameter (or a function) is reconstructible (and the parameter on the dual of a poset is of interest), then the *dual parameter (function)* is also reconstructible. (The number of minimal and the number of maximal elements of a poset are an important example of dual parameters.)

2. Given a deck, we can construct the *comparability graph deck* $G(P_1), G(P_2), \ldots, G(P_n)$. Clearly, if P is a preimage of $P_1, P_2, \ldots, P_n$, then $G(P)$ is a preimage of $G(P_1), G(P_2), \ldots, G(P_n)$. This enables us to use the reconstructibility of certain classes of graphs and graph parameters for the reconstruction of posets.

The reconstructibility of classes of graphs is one of the main topics in graph reconstruction theory. We shall follow this line of research for posets too. In what follows, we review the definitions of some important classes of posets. For more details we refer the reader to [5, 18, 23] for poset-theoretic notions and to [1, 10, 12] for graph-theoretic notions.

Let $P = (V(P), \prec_P)$ be a poset:

P is said to be *disconnected* (respectively, *connected*) if its HASSE-diagram is a disconnected (respectively, connected) graph.

P is said to be *series-parallel* iff it has no "N" (see figure 4 (a)) as subposet, iff its comparability graph is a cograph (i.e. it has no induced path on 4 vertices as an induced subgraph). The class of series-parallel posets is the smallest class of posets containing the one element poset and being closed under parallel composition and series composition.

P is said to be *N-free* if it has no "N" as covering subposet (see figure 4 (b)).

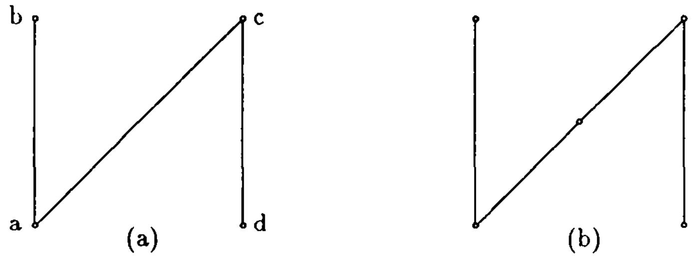

Figure 4: (a) The HASSE-diagram of the poset "N" ; (b) an N-free poset

P is said to be an *interval order* if it can be represented by assigning a real interval $I_x = [a_x, b_x]$ to each element $x \in V(P)$, such that $x \prec_P y$ iff $b_x < a_y$ for all $x, y \in V(P)$. Equivalently, P is an interval order iff the complement of its comparability graph is an interval graph, iff its sets of predecessors $Pred(x) = \{y \in V(P) : y \prec_P x\}$ (respectively, successors $Succ(x) = \{y \in V(P) : x \prec_P y\}$) are linearly ordered by inclusion.

P is said to be a *bipartite* poset if its height is at most one.

Let S be a subset of $V(P)$. An element $y \in V(P)$ is called an *upper bound* for S in P, if $s \prec_P y$ for every $s \in S$. Similarly, y is a *lower bound* for S if $y \prec_P s$ for every $s \in S$. Moreover, S has a *supremum* in P, if the set of all its upper bounds has a unique lower bound in P. This unique lower bound is said to be the *supremum* of S. Similarly, S has an *infimum* in P if the set of all its lower bounds has a unique upper bound in P.

P is said to be a *lattice*, if every nonempty two-element subset $S \subseteq V(P)$ has both an infimum and a supremum in P. Lattices have plenty of nice algebraic properties and are also widely related to posets (cf. [2, 19]). The only property we need is that finite lattices always have a unique minimal and a unique maximal element.

3 Direct consequences of known results for graphs

First we summarize some of the known results on the reconstructibility of graph parameters and classes of graphs and then give, for some of them, an interpretation for poset reconstruction. For definitions and more details we refer to one of the excellent surveys on reconstruction [3, 4, 13, 20] .

The following classes of graphs are reconstructible: trees, disconnected graphs, regular graphs [15], unit interval graphs and threshold graphs [26], separable graphs with no pendant vertex and outerplanar graphs.

The survey of Bondy [3] emphasizes special techniques developed to attack the reconstruction problem. A well-known and very useful one is Kelly's Lemma, established in his doctoral Thesis [15].

Lemma 3.1 **[15]** *For any two graphs $F = (V(F), E(F))$ and $G = (V(G), E(G))$ with $| V(F) | < | V(G) |$ is the number of subgraphs of G isomorphic to F reconstructible. Furthermore, the number of subgraphs of G that are isomorphic to F and contain a given vertex v is reconstructible.*

Among others, this establishes an easy proof that the maximum size of a clique (respectively, independent set) of a graph G is reconstructible.

Now, simply using the fact that the deck of the poset creates a deck of the comparability graph of the poset, one gets:

Theorem 3.2 *The following parameters of a poset P are reconstructible:*

1. *the number of elements $| V(P) |$,*
2. *the number of edges of the comparability graph $| E(G(P)) |$,*
3. *the degree of x in the comparability graph $G(P)$ from the card $G(P - \{x\})$,*
4. *the degree sequence of the comparability graph,*
5. *$h(P)$ and $w(P)$.*

Looking for the reconstruction of perfect graphs, von Rimscha [26] obtained among other results the following:

Theorem 3.3 **[26]** *Permutation graphs, interval graphs and unit interval graphs are recognizable.*

Since cographs are reconstructible [6] and as cographs are exactly the comparability graphs of series-parallel posets, permutation graphs are exactly the comparability graphs of posets of dimension two and unit interval graphs are exactly the comparability graphs of semi-orders, with the above Theorem 3.3, we get:

Theorem 3.4 *Posets of dimension two, interval orders, semi-orders and series-parallel posets are recognizable.*

4 The Kelly Lemma and the reconstructibility of certain classes of posets

The following statement is a translation of Lemma 3.1 , the well-known Kelly Lemma [15], to the reconstruction of posets.

Lemma 4.1 *Let P and Q be two posets with $|V(Q)|<|V(P)|$, then the number $s(Q,P)$ of subposets of P isomorphic to Q is reconstructible. Furthermore, the number of subposets of P that are isomorphic to Q and contain a given vertex v is reconstructible.*

Proof. Each subposet of P isomorphic to Q occurs in exactly $|V(P)-V(Q)|$ one-element-deleted subposets of P, thus we get: $s(Q,P)\cdot|V(P)-V(Q)|=\sum_{x\in V(P)} s(Q,P-\{x\})$. Since the second part of the equality is clearly computable from the deck, the result follows.
The number of subposets of P containing the vertex v, being isomorphic to Q, is simply: $s(Q,P)-s(Q,P-\{v\})$. q.e.d.

Remark:

1. The number of occurencies of a poset Q in a poset P, does not correspond with the number of occurencies of the corresponding graph $G(Q)$ in $G(P)$.
2. An interesting poset version of Kelly's Lemma would occur when *subposet* could be replaced by *covering subposet* in Lemma 4.1 . Although such a lemma looks like a good deal, a general counting formula seems not so easy to obtain.

Similar to the situation for graphs, Lemma 4.1 allows to attack disconnected posets. Assume that a connected component of a poset P is the suborder $P|_S$ of P, where S is a set of vertices of a connected component of its comparability graph $G(P)$. Then, the proof of Theorem 4.2 can be given analogously (cf. the proof given for the poset version of the Kelly Lemma) to an original one for graphs and is therefore omitted here.

Theorem 4.2 *Disconnected posets are reconstructible.*

Here appears some kind of tool provided by the well-structured binary relation which are additionally posets. Indeed, taking into account the level-structure (structure widely used in the following) allows us to obtain, quite naturally, an interesting reconstructible class of posets:

Theorem 4.3 *Posets with unique minimal element are reconstructible.*

Proof. If $|MIN(P)|=1$, at least $n-1$ cards have a unique minimal element.
If $|MIN(P)|>2$, there is no card with unique minimal element.
If $|MIN(P)|=2$, then at least $n-2$ cards have more than one minimal element. Consequently, posets with unique minimal element are recognizable.
We are going to show how to construct a poset P with a unique minimal element from its deck: First, P is a chain if and only if all cards are chains. Otherwise, consider a deck with a minimal lowest level with more than one element. This poset is clearly isomorphic to $P-\{x\}$ where x is the minimal element of P. Consequently, we can reconstruct P. q.e.d.

As immediate consequence we obtain the reconstructibility of lattices:

Corollary 4.4 *Lattices are reconstructible.*

By duality, we get:

Corollary 4.5 *Posets with unique maximal element are reconstructible.*

5 The reconstructibility of the number of minimal and the number of maximal elements of a poset

W.l.o.g. we may assume that no element of (a preimage) P of our given deck of cards $P_1, P_2, P_3, \ldots, P_n$, with $n = |P| \geq 4$, is both minimal and maximal element in P. Otherwise, P would have a disconnected HASSE-diagram and therefore P would be reconstructible.

Definition 1 *Let $MIN(P)$ and $MAX(P)$ be the set of all minimal and maximal elements, respectively, of a poset P. Let $P_1, P_2, P_3, \ldots, P_n$ be the deck of P, we define:*

$$\widetilde{min} =_{Df} min_{1 \leq i \leq n} \, |MIN(P_i)|$$

and

$$\widetilde{max} =_{Df} min_{1 \leq i \leq n} \, |MAX(P_i)|$$

Some of the quite immediate but nevertheless fundamental relations between the number of maximal (respectively, minimal) elements of a poset and its cards are stated in the following:

Lemma 5.1 *Let $P = (V(P), \prec_P)$ be a connected poset with deck $P_1, P_2, P_3, \ldots, P_n$, then:*

(i) *For every $i \in \{1, 2, \ldots, n\}$ holds $|MIN(P)| = |MIN(P_i)|$ or $|MAX(P)| = |MAX(P_i)|$. Moreover, there are $i, j \in \{1, 2, \ldots, n\}$ such that $|MIN(P)| = |MIN(P_i)|$ and $|MAX(P)| = |MAX(P_j)|$.*

(ii) *For every $i \in \{1, 2, \ldots, n\}$ holds $|MIN(P)| - 1 \leq |MIN(P_i)|$ and $|MAX(P)| - 1 \leq |MAX(P_i)|$.*

(iii) *We have $|MIN(P)| \in \{\widetilde{min}, \widetilde{min} + 1\}$ and $|MAX(P)| \in \{\widetilde{max}, \widetilde{max} + 1\}$.*

(iv) *If P is a nonbipartite poset, i.e. $h(P) > 1$, then there exists an $i \in \{1, 2, \ldots, n\}$ such that $|MIN(P)| = |MIN(P_i)|$ and $|MAX(P)| = |MAX(P_i)|$.*

Proof. **(i)** follows from the fact that deleting a minimal (respectively, maximal) element in a poset does not change the number of maximal (respectively, minimal) elements.

(ii) follows from the fact that if we delete a minimal (respectively, maximal) element of a poset then all the others still remain minimal (respectively, maximal) in the one-element-deleted subposet.

(iii) follows immediately from **(ii)**.

(iv) follows from the fact that deleting an element which is neither minimal nor maximal does not change any of the parameters. q.e.d.

We are now able to prove the reconstructibility of the number of maximal and minimal elements. This result has already been announced by S.K. Das in his doctoral Thesis [9], but it is, up to our knowledge, unpublished.

Theorem 5.2 *For every poset P, with $|V(P)| > 3$, $|MIN(P)|$ and $|MAX(P)|$ can be determined from the deck of P. Consequently, the number of minimal elements and the number of maximal elements is reconstructible.*

Proof. Since disconnected posets are reconstructible, by Theorem 4.2, we assume P to be connected. We show how to determine $|MAX(P)|$: we know, by Lemma 5.1, that $|MAX(P)| \in \{\widetilde{max}, \widetilde{max}+1\}$ and that there is an $i \in \{1,2,\ldots,n\}$ such that $|MAX(P_i)| = |MAX(P)|$, hence if among all cards only one of the values $\widetilde{max}$ or $\widetilde{max}+1$ occurs, then this is the value of $|MAX(P)|$. Let us denote by A_0 (respectively, A_1) the number of cards with $|MAX(P_i)| = \widetilde{max}$ (respectively, $|MAX(P_i)| = \widetilde{max}+1$). Since cards with a correct number of maximal element must appear at least $n - |MAX(P)|$ times, the following holds:

1. Assuming $|MAX(P)| = \widetilde{max}$, then $A_0 \geq n - \widetilde{max}$ holds.
2. Assuming $|MAX(P)| = \widetilde{max}+1$, then $A_1 \geq n - (\widetilde{max}+1)$ holds.

Thus, if one of these inequalities is not fulfilled, we got the value of $|MAX(P)|$. Otherwise, we may assume that $A_0 \geq n - \widetilde{max}$ and $A_1 \geq n - (\widetilde{max}+1)$ hold. In order to conclude the proof, we are going to show (assuming $n > 3$) that in this remaining case $|MAX(P)| = \widetilde{max}+1$ holds. Assume not, i.e. $|MAX(P)| = \widetilde{max}$, thus the only way to get a card with $\widetilde{max}+1$ maximal elements is by the deletion of a maximal element of P having 2 private predecessors. It is then clear that $n \geq 2 \cdot A_1 + |MAX(P)|$, therefore $n \geq 2 \cdot n - \widetilde{max} - 2$, thus $\widetilde{max} \geq n-2$. But, since there exists a card with $\widetilde{max}+1$ elements, we have $\widetilde{max} \leq n-2$. Thus, the only available value would be $\widetilde{max} = n-2$. Since P is connected and there is a least one maximal element of P, having 2 private predecessors, this is only possible for $n = 3$ (see figure 1).
The proof for $|MIN(P)|$ goes, by duality, along the same lines. q.e.d.

Following our proof, we get immediately:

Corollary 5.3 *For every connected poset $P = (V(P), \prec_P)$, the number of its maximal and minimal elements can be computed in time $O(|V(P)|^2)$ from its deck.*

6 Ideal-size and filter-size sequences. Minimal and maximal cards. The level-structure of the HASSE-diagram

The following approach uses special properties of posets and relies mostly on the ideal and filter notions. The concept of ideal-size and filter-size sequences, with all the properties given here, is powerful enough to enable the determination of a minimal card from the deck.

Definition 2 *For any $x \in V(P)$, $\mathcal{I}(x) =_{Df} \{y \in V(P) : y \prec_P x\} \cup \{x\}$ is said to be the ideal of x in P, and $\mathcal{F}(x) =_{Df} \{y \in V(P) : x \prec_P y\} \cup \{x\}$ is said to be the filter of x in P.*

$i(x) =_{Df} |\mathcal{I}(x)|$ and $f(x) =_{Df} |\mathcal{F}(x)|$ are said to be the ideal-size and the filter-size of an element $x \in V(P)$, respectively.

From the definition it follows immediately that for every $x, y \in V(P)$ holds: $y \prec_P x \Rightarrow i(y) < i(x)$ and $y \prec_P x \Rightarrow f(y) > f(x)$.

It turns out that the study of properties of the ideal-size sequence of the maximal elements of a poset and how it changes if one element of the poset is deleted is important. Throughout the consideration of ideal-size and filter-size sequences, we always assume that these sequences of natural numbers are ordered monotone decreasing.

Let $x_1, x_2, \ldots, x_r$ be the maximal elements of a poset P such that $i(x_1) \geq i(x_2) \geq \ldots \geq i(x_r)$ holds. Let $P' = P - \{z\}$ be a one-element-deleted subposet of P with maximal elements $y_1, y_2, \ldots, y_s$ such that $i(y_1) \geq i(y_2) \geq \ldots \geq i(y_s)$ holds. Then, the following facts are immediate consequences from the definition:

1. If z is not a maximal element of P, then $r = s$ and $i(x_j) - i(y_j) \in \{0, 1\}$ hold for every $j \in \{1, 2, \ldots, r\}$.

2. If $z = x_k$ is a maximal element of P, then $r - 1 \leq s$ and the sequence $i(x_1), i(x_2), \ldots, i(x_{k-1}), i(x_{k+1}), \ldots, i(x_r)$ is a subsequence of the sequence $i(y_1), i(y_2), \ldots, i(y_s)$. For every maximal element y_j of P', not belonging to $MAX(P)$, we also have $i(y_j) < i(x_k)$. Furthermore, $i(x_k)$ is exactly one more than the degree of x_k in the comparability graph of P, which can be determined from the card $P - \{x_k\}$.

In general, it is not obvious to detect a card of the deck which is definitely created by deleting a maximal element for any preimage of the deck. It also seems hard to find out which of the maximal elements of a card are maximal in P. We shall give partial solutions to these questions in the paper.

(Note that for the deck created by the posets of figure 1 it is impossible to reconstruct the number of maximal elements and no card of the deck is created from both reconstructions by deleting a maximal element!)

Similar facts, as the ones stated above for maximal elements, hold for minimal elements.

Theorem 6.1 *The ideal-size sequence of the maximal elements of a poset P is reconstructible.*

Proof. If there is a card P_k with $|MAX(P_k)| = |MAX(P)| - 1$, then we have $P_k = P - \{x_k\}$, where x_k is a maximal element of P. Thus, it follows immediately from the above facts, that the ideal-size sequence of P_k is exactly the one of P without the missing entry $i(x_k) = deg_{G(P)}(x_k) + 1$, which can be determined from the deck, by Theorem 3.2.

Otherwise, for every card P_k of the deck holds $|MAX(P_k)| \geq |MAX(P)|$. Therefore every $x_k \in MAX(P)$ has a private predecessor, say $p(x_k)$, with the property:

$$p(x_k) \in \mathcal{I}(x_k) - \bigcup_{x \in MAX(P) - \{x_k\}} \mathcal{I}(x).$$

Now, let $x_1, x_2, \ldots, x_r$ be the maximal elements of the poset P such that $i(x_1) \geq i(x_2) \geq \ldots \geq i(x_r)$ holds. Then $P - \{p(x_r)\}$ has $i(x_1), i(x_2), \ldots i(x_{r-1}), i(x_r) - 1$ as ideal-size sequence of the maximal elements. Hence, an ordering of the ideal-size sequences of the maximal elements of all the cards of the deck in quasi-lexicographic order (a sequence is smaller than another one if the first entry, differing from the corresponding one, is smaller) will produce as largest sequence $i(x_1), i(x_2), \ldots i(x_{r-1}), i(x_r) - 1$. Thus, we can reconstruct the ideal-size sequence of the maximal elements of P. **q.e.d.**

By duality, we have:

Corollary 6.2 *The filter-size sequence of the minimal elements of a poset is reconstructible.*

Definition 3 *A card of the deck of P is said to be minimal (respectively, maximal) if this card, might have been created by deleting a minimal (respectively, maximal) element, for any reconstruction of P.*

The knowledge of the filter-size sequence of the minimal elements of a poset allows the determination of a minimal card. By the way, provide a powerful tool toward the reconstruction of posets.

Theorem 6.3 *For every poset P, one minimal card can be determined. Furthermore, its minimal elements which are not minimal element in P can be determined too.*

Proof. If there is a card P_k with $|MIN(P_k)| = |MIN(P)| - 1$, then it is a minimal card and $MIN(P_k) - MIN(P) = \emptyset$.

Otherwise, for every card P_i of the deck holds $|MIN(P_i)| \geq |MIN(P)|$. By Corollary 6.2, the filter-size sequence of the minimal elements of P can be reconstructed from the deck of P. We may assume it to be $f(v_1) \geq f(v_2) \geq \ldots \geq f(v_s)$, where $MIN(P) = \{v_1, v_2, \ldots, v_s\}$.
Let P_k be a card of P, with $MIN(P_k) = \{a_1, a_2, \ldots, a_t\}$, $f(a_1) \geq f(a_2) \geq \ldots \geq f(a_t)$ such that for every $i \in \{1, 2, \ldots, s-1\}$, we have $f(a_i) = f(v_i)$ and $f(a_{s-1}) > f(a_s)$ (one can show that such a card exists using similar arguments as for the proof of Corollary 6.1), thus for $i \in \{1, 2, \ldots, s-1\}$, we have $a_i \in MIN(P)$. Then the following holds:

1. Assume that $t > s$: then P_k is a minimal card and $\{a_s, a_{s+1}, \ldots, a_t\}$ are not minimal elements of P. Thus $MIN(P_k) - MIN(P) = \{a_s, a_{s+1}, \ldots, a_t\}$.
2. Now assume that $t = s$:
   (a) Assume that $f(a_s) < f(v_s) - 1$: then P_k is a minimal card and a_s is not a minimal element of P. Thus $MIN(P_k) - MIN(P) = \{a_s\}$.
   (b) Now assume that $f(a_s) = f(v_s) - 1$ and that we did not already find a minimal card by foregoing cases: Take all cards $\{P_i\}_{i \in I}$ with filter-size sequence $f(v_1), f(v_2), \ldots, f(v_{s-1})$, $f(v_s) - 1$ and notice that all these cards arise by deleting an element $z \in \mathcal{F}(v_s)$ such that $z \notin \bigcup_{j=1}^{s-1} \mathcal{F}(v_j)$. We consider the level-structure of all these cards P_i (i.e. the sequence $|H_0(P_i)|, |H_1(P_i)|, \ldots, |H_{h(P)}(P_i)|$). Here, the maximal level-structure (sequence) sorted by lexicographic-order occurs on a card, which we call P_k where the poset is isomorphic to $P - \{v_s\}$. Hence P_k is a minimal card and $MIN(P_k) - MIN(P) = \{a_s\}$.

q.e.d.

Remark: The described construction does not give us a reconstruction of the poset, since we do not know the immediate successors of v_s, which did not bump down one level from $H_1(P)$ to $H_0(P - \{v_s\})$.

From Theorem 6.3 we can deduce the level-structure of any poset. This result is one of those announced by S.K. Das in his doctoral Thesis [9].

Corollary 6.4 *The level-structure ($|H_0(P)|, |H_1(P)|, \ldots, |H_{h(P)}(P)|$) of a poset P is reconstructible.*

Proof. Let P_k be the card determined by Theorem 6.3 and let $U = MIN(P_k) - MIN(P)$. Then, the level-structure of P is the one of the poset P' where $P'|_{V(P_k)} = P_k$, $V(P') - V(P_k) = \{x\}$ and $\mathcal{F}(x) = \bigcup_{y \in U} \mathcal{F}(y)$. (Note that P' is in general not a reconstruction of the deck!) q.e.d.

Corollary 6.5 *The filter-size sequence of any fixed level of the HASSE-diagram of a poset P is reconstructible.*

Proof. Let P_k be the card determined by Theorem 6.3 and let $U = MIN(P_k) - MIN(P)$. Let P' be a poset such that $P'|_{V(P_k)} = P_k$, $V(P') - V(P_k) = \{x\}$ and $\mathcal{F}(x) = \bigcup_{y \in U} \mathcal{F}(y)$. There is a bijection $\sigma : V(P) \longrightarrow V(P')$ such that for every $v \in V(P)$ $rank(v, P) = rank(\sigma(v), P')$. Since the poset P_k is isomorphic to $P - \{v_s\}$ (where $v_s \in MIN(P)$), all elements of the card P_k have the correct filter size. Thus from P' we get the filter-size sequence of all levels assuming that we add $f(x_k) = deg_{G(P)}(x_k) + 1$ to the level H_0 (where $deg_{G(P)}(x_k)$ is determined by the card P_k). q.e.d.

By duality, we get:

Corollary 6.6 *For every poset P one maximal card can be determined.*

Corollary 6.7 *The ideal-size sequence of any fixed level of the HASSE-diagram of a poset P is reconstructible.*

Proof. By Corollary 6.6, we can find a maximal card in the deck of P, say $P - \{x\}$. By Corollary 6.4, we know the level structure of P, hence we know $rank(x, P)$. Thus, the ideal size sequence of any fixed level of $P - \{x\}$ is the one of P, assuming we add $deg_{G(P)}(x_k) + 1$ to the sequence of level $rank(x, P)$. q.e.d.

Remark: By Corollary 6.5 , we can determine for any poset P the level distribution of the maximal elements from its deck.

7 The number of edges of the HASSE-diagram is reconstructible

In this section, a first step towards a Kelly Lemma for covering subposet, in the special case where the covering subposet is a chain with two elements, is given. Moreover, this result is somehow interesting as the number of edges in the HASSE-diagram is not a comparability invariant.

Theorem 7.1 *The number of edges in the HASSE-diagram is reconstructible.*

Proof. For a poset P, we analyze the overall number of edges of the HASSE-diagram on all its cards, i.e. $\sum_{i=1}^{n} |E(H_i)|$, where H_i is the HASSE-diagram of the card P_i.
Only two types of edges occur on cards of the deck.
Case 1: *Original edges*
Every edges of $H(P)$ occur on exactly $n - 2$ cards (analogously to graph reconstruction).
Case 2: *2-transitive edges*
Every 2-transitive edge of P occurs in exactly one HASSE-diagram of the deck of P. By Theorem 6.3, we can determine a minimal card in the deck of P, say $P - \{x\}$. Then all 2-transitive edges of P, not incident to x, are 2-transitive edges of $P - \{x\}$.

Let $|\mathcal{T}(P - \{x\})|$ be the number of 2-transitive edges on $P - \{x\}$. Let $|\mathcal{T}(x)|$ be the number of 2-transitive edges incident to the minimal element x. Now we have the following formula:

$$\sum_{i=1}^{n} |E(H_i)| = (n-2)\,|E(H)| + |\mathcal{T}(P - \{x\})| + |\mathcal{T}(x)|\,.$$

Consequently, we get:

$$|E(H)| = \frac{\sum_{i=1}^{n} |E(H_i)| - |\mathcal{T}(P - \{x\})| - |\mathcal{T}(x)|}{n-2}.$$

$\sum_{i=1}^{n} |E(H_i)|$ can be determined from the deck and $|\mathcal{T}(P - \{x\})|$ can easily be determined on $P - \{x\}$. Thus, by our formula, if $|\mathcal{T}(x)| \leq n - 3$, there is only one natural number for the left-hand side. Since $|\mathcal{T}(x)| \leq deg_{G(P)}(x) - 1$ holds, this is fulfilled if $deg_{G(P)}(x) \leq n - 2$. Hence, we remain with the case $deg_{G(P)}(x) = n - 1$. This implies that x is the unique minimal element of P, and we already know that posets with unique minimal element are reconstructible by Theorem 4.3. q.e.d.

8 Interval orders and N-free orders

Interval orders are known to be recognizable by Theorem 3.4. They are strongly structured by the inclusion relation between successor sets and predecessor sets, respectively, of their elements. This successor set property which supports a polynomial time algorithm for the isomorphism problem [18] is widely used here for proving the reconstructibility of such orders.

Lemma 8.1 *Let P be an interval order, then for every $x, y \in V(P)$ holds:*

$$i(x) \leq i(y) \Longleftrightarrow Pred(x) \subseteq Pred(y) \text{ and } i(x) = i(y) \Longleftrightarrow Pred(x) = Pred(y)$$

$$f(x) \leq f(y) \Longleftrightarrow Succ(x) \subseteq Succ(y) \text{ and } f(x) = f(y) \Longleftrightarrow Succ(x) = Succ(y)$$

Proof. The equivalences follow immediately from the fact, that the successor sets (respectively, predecessor sets) ordered by inclusion form a chain in interval orders. q.e.d.

Theorem 8.2 *Interval orders are reconstructible.*

Proof. Interval orders are recognizable by Theorem 3.4, thus it is enough to show weak reconstructibility of interval orders here.

By Section 4, we may assume P to be a connected interval order with at least 2 maximal and with at least 2 minimal elements. W.l.o.g. let $x_1, x_2, \ldots, x_s$, $s \geq 2$, be the maximal elements of P such that $i(x_1) \geq i(x_2) \geq \ldots \geq i(x_s)$. Since P is an interval order, we have $z \in \mathcal{I}(x_1)$ for every $z \in V(P) - MAX(P)$.

Suppose we had determined from the deck of P a card isomorphic to $P - \{x_1\}$. Let S be a subset of $MAX(P - \{x_1\})$ having ideal-size sequence $i(x_2), i(x_3), \ldots, i(x_s)$. Then, using Lemma 8.1, P can be reconstructed in the following way: A new element, say x_1, is added to this card such that x_1 has no successor in the new poset and such that the predecessor set of x_1 in the new poset is exactly the union of the ideals of all elements in $MAX(P - \{x_1\}) - S$ and the predecessor sets of all elements in S. With other words, x_1 is added to the card such that the set of maximal elements in the new poset is exactly $S \cup \{x_1\}$, while all non-maximal elements belong to the predecessor set of x_1. Clearly, the constructed poset is the only reconstruction of the deck under consideration, i.e. P itself.

Thus its remains to show, how to determine from the deck of P, a card isomorphic to $P - \{x_1\}$.

Assuming $i(x_1) = i(x_2)$, all cards $P - \{x\}$ with $x \in MAX(P)$ have the property $|MAX(P - \{x\})| = |MAX(P)| - 1$. Hence, they can easily be determined. Among these cards, all cards $P - \{x\}$ with $deg_{G(P)}(x) = |V(P) - MAX(P)|$ are isomorphic to $P - \{x_1\}$.

Thus, we may assume for the remaining part of the proof that $i(x_1) > i(x_2)$ holds:

Case 1: *x_1 has at least two private predecessors in P.*

The only card $P - \{x\}$ with $|MAX(P - \{x\})| > |MAX(P)|$ is the card $P - \{x_1\}$.

Case 2: *x_1 has exactly one private predecessor in P, say $p(x_1)$.*

Case 2.1: $i(p(x_1)) \neq i(x_1) - 1$.

$P - \{x_1\}$ is the only card $P - \{x\}$ such that $|MAX(P - \{x\})| = |MAX(P)|$ and $i(x_2), i(x_3), \ldots, i(x_s)$ is a subsequence of the ideal-size sequence of its maximal elements such that the remaining ideal-size differs from $i(x_1) - 1$.

Case 2.1: $i(p(x_1)) = i(x_1) - 1$.

The only immediate predecessor of x_1 is $p(x_1)$ and for every $z \in V(P) - MAX(P)$, we have $z \in \mathcal{I}(p(x_1))$. Moreover, $P - \{x_1\}$ has the following ideal-size sequence of the maximal elements:

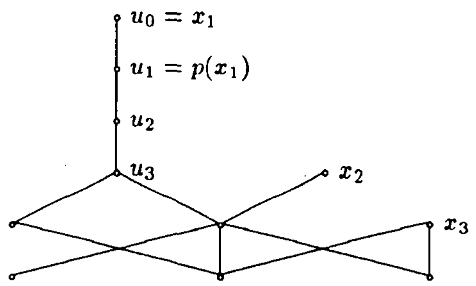

Figure 5: An interval order with a hair of length 3

$i(x_1)-1, i(x_2), \ldots, i(x_s)$. On the other side, if $P-\{z\}$ is a card with exactly this ideal-size sequence of the maximal elements, then necessarily holds:

$$z \in \mathcal{I}(x_1) - \bigcup_{i=2}^{s} \mathcal{I}(x_i) = \mathcal{I}(x_1) - \mathcal{I}(x_2).$$

Since x_1 has only the immediate predecessor $p(x_1)$, there are elements $u_0 = x_1, u_1 = p(x_1), u_2, \ldots, u_l$ of $V(P)$ with $l \geq 1$ such that:

1. for every $i \in \{1, 2, \ldots, l\}$, u_i is a private predecessor of u_{i-1},
2. for every $i \in \{1, 2, \ldots, l\}$, u_i is the only immediate predecessor of u_{i-1} and
3. u_l has (either none or) more than two immediate predecessors.

Hence, $deg_{H(P)}(u_0) = 1$, $deg_{H(P)}(u_i) = 2$ for every $i \in \{1, 2, \ldots, l-1\}$ and $deg_{H(P)}(u_l) > 2$ hold. (If u_l has no immediate predecessor and $deg_{H(P)}(u_l) = 1$, then the hair is a connected component of P which is a chain, thus this case is of no interest for the reconstruction problem.) We call such a covering subposet a *hair* of P and l is said to be the *length of the hair*. (see figure 5).

Finally, a card $P - \{u_i\}$ is isomorphic to $P - \{x_1\}$, if—among all the cards with $i(x_1) - 1$, $i(x_2), \ldots, i(x_s)$ as ideal-size sequence of the maximal elements—its hair containing the element u_0 with $i(u_0) = i(x_1) - 1$ is of smallest length. (Note that on all these cards there is exactly one element with ideal-size $i(x_1) - 1$.)

Consequently, we are always able to determine a card isomorphic to $P - \{x_1\}$ and to reconstruct P uniquely, as shown above. **q.e.d.**

The recognizability of N-free orders provides an interesting light on the reconstruction problem since this class of posets is not hereditary.

Theorem 8.3 *N-free orders are recognizable.*

Proof. Let $P_1, P_2, \ldots, P_n$ be the deck of an N-free poset P. We are going to show that every reconstruction P' of P is also an N-free poset. W.l.o.g. we can assume that P is a connected poset with at least 2 maximal and 2 minimal elements. Otherwise, P is reconstructible (see Section 4).
Fact 1: The deletion of any maximal or minimal element of an N-free poset Q yield an N-free one-element-deleted subposet of Q.
Thus if Q is a connected N-free poset, it has at least $|MAX(Q)| + |MIN(Q)|$ N-free cards in its deck.

Fact 2: If a poset Q contains an "N" as covering subposet then this "N" is a covering subposet of exactly $|V(Q)|-4$ one-element-deleted subposets of Q.
Thus if Q is not an N-free poset, it has at least $|V(Q)|-4$ non N-free cards in its deck. Moreover, if Q has at least 2 distinct covering subposets yielding an "N" , then Q has at least $|V(Q)|-3$ non N-free cards in its deck

Consequently, if P has at least 3 minimal or at least 3 maximal elements then at most $|V(P)|-5$ cards of its deck are non N-free poset and thus none of its reconstruction can be a non N-free poset. Hence, we may assume that P has exactly 2 minimal and 2 maximal elements and also that P has exactly $|V(P)|-4$ cards (those obtained by the deletion of neither a minimal nor a maximal element of P) which are non N-free posets. Otherwise, we know that all reconstructions of P are N-free posets.

Let P' be a reconstruction of such a poset P. We are going to show that P' can only be an N-free poset: Assume that P' is not an N-free poset, then P' has exactly one "N" as covering subposet. Moreover, by Theorem 6.3 and Corollary 6.6, we can determine one minimal and one maximal card from the deck of P, which both have to be N-free. Since P is N-free, the "N" of P' must share at least one maximal and one minimal element of P'. Let us try to construct such a poset P', meeting all these conditions.

Since the poset "N" is reconstructible, we may assume that $|V(P)|\geq 5$. Moreover, if $MIN(P')\cup MAX(P')$ induce the "N" then P' has the structure given in figure 6 and therefore contains more than one "N" .

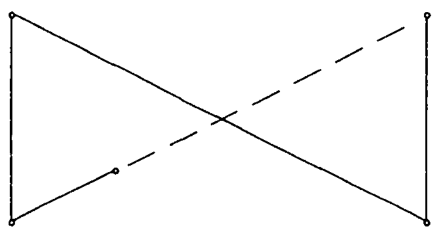

Figure 6: $MIN(P')\cup MAX(P')$ induce the "N"

Consequently, we may assume that the "N" contains an element z which is neither minimal nor maximal. Let the only "N" of P' be labeled as in figure 4 (a).

Case 1: $z=d$

Since $z\notin MIN(P')$, there exists an element w covered by z in P' (see figure 7 (a.1)):

Notice that w is not a predecessor of a because $a\in MIN(P')$ and that w is not a successor of a, otherwise z would be a successor of a in P'. If (w,c) is a 2-transitive edge of P', then $P'-\{z\}$ has an "N" , thus we get more than $|V(P)|-4$ non N-free cards in the deck of P'. Hence, P' cannot be a reconstruction of P.

If (w,c) is not a 2-transitive edge of P', then $|V(P)|\geq 6$ and there is $u\in V(P')-\{a,b,c,z,w\}$ such that u is covered by c and u is a successor of w in P' (see figure 7 (a.2)). Note that u cannot be a successor of b in P', otherwise, c would be a successor of b in P'.
If u is not a predecessor of b then $\{u,c,a,b\}$ induces a second "N" in P'. If u is a predecessor of b, then either u is covered by b then $\{b,u,c,z\}$ induces a second "N" in P' or $|V(P)|\geq 7$ and there is $v\in V(P')-\{a,b,c,z,w,u\}$ such that u is covered by v and v is a predecessor of b, hence $\{v,u,c,z\}$ induces a second "N" in P'.
Hence, if $z=d$, there is no poset P' having only one "N" , containing a maximal and a minimal

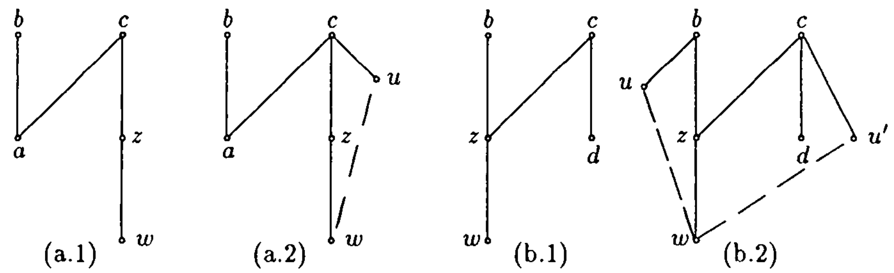

Figure 7: Case 1: $z = d$; Case 2: $z = a$

element with $|V(P)|-4$ non N-free cards in its deck. Thus P' must be N-free.

Case 2: $z = a$

Since $z \notin MIN(P')$, there exists an element w covered by z in P' (see figure 7 (b.1)):
Notice that w is not a predecessor of d because $d \in MIN(P')$ and that w is not a successor of d, otherwise, b would be a successor of d in P'.
If (w, b) and (w, c) are 2-transitive edges of P', then $P' - \{z\}$ has an "N" , thus we get more than $|V(P)|-4$ non N-free cards in the deck of P'. Hence, P' cannot be a reconstruction of P.
If (w, b) is not a 2-transitive edge of P', then $|V(P)| \geq 6$ and there exists an element $u \in V(P') - \{d, b, z, w, c\}$ such that u is covered by b and u is a successor of w in P' (see figure 7 (b.2)). Note that u cannot be a successor of c in P', otherwise, b would be a successor of d in P'. If u is not a predecessor of c, then $\{u, b, z, c\}$ induces a second "N" in P'. If u is a predecessor of c, then either u is covered by c and $\{d, c, u, b\}$ induces a second "N" in P' or $|V(P)| \geq 7$ and there exists $v \in V(P') - \{w, z, u, b, c, d\}$ such that u is covered by v and v is a predecessor of c, consequently, $\{v, u, b, z\}$ induce a second "N" in P'.
If (w, c) is not a 2-transitive edge of P', then $|V(P)| \geq 6$ and there exists an element $u' \in V(P') - \{w, z, b, c, d\}$ such that u' is covered by c and u' is a successor of w (see figure 7 (b.2)). Note that u' cannot be a successor of b in P', otherwise, c would be a successor of b in P'. If u' is not a predecessor of b, then $\{u', c, z, b\}$ induces a second "N" in P'. If u' is a predecessor of b, then either u' is covered by b and $\{d, c, u', b\}$ induces a second "N" in P' or u' is not covered by b, thus $|V(P)| \geq 7$ and there exists $v \in V(P') - \{d, b, z, w, c, u'\}$ such that u' is covered by v and v is a predecessor of b in P', consequently, $\{v, u', c, z\}$ induces a second "N" in P'. Hence, if $z = a$, there is no poset P', having exactly one "N" , which contains a maximal and a minimal element, with $|V(P)|-4$ non N-free cards in its deck. Thus P' must be N-free.

The remaining **Case 3:** $z = b$ and **Case 4:** $z = c$ follow by duality. q.e.d.

9 Open Questions

Here are some of the—from our point of view—most interesting open questions in the field of poset reconstruction:

1. Is for every poset P the number of occurencies of a certain covering subposet T, with $|V(T)| < |V(P)|$, the same for all reconstructions of P ? (We would call this statement a Kelly Lemma for covering subposets.)
2. Is the reconstructibility a comparability invariant? This means, if a poset is reconstructible, then all posets with the same comparability graph are reconstructible.

3. Can one determine more than one, or even all minimal (respectively, maximal) cards from the deck of any poset?

The last question is strongly related to the following amazing question of B. Sands [25]:

> Is every finite poset P uniquely determined—up to isomorphism—by its collection of (unlabelled) one-element-deleted subposets $\langle P - \{x\} : x \in MAX(P) \rangle$?

A positive answer would have strong consequences to the reconstructibility of certain classes of posets.
Finally, here is our favourite conjecture:

Conjecture: *If Q is a reconstruction of P not isomorphic to P, then $G(P) \neq G(Q)$.*

This statement and the reconstructibility of comparability graphs, which is indeed—up to now—an open question, would imply that the poset reconstruction conjecture is true.

Acknowledgement

The first author wishes to thank Michel Habib for hospitality during the time at Montpellier, when this research started.

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Imprimé en France
par
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ISSN 0249 - 6399